A RESULT ON REARRANGEMENTS*

BY

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ABSTRACT

Circular symmetry is defined for ordered sets of n real numbers: $(y)=(y_1,\cdots,y_n)$. Let $f(x)$ be non-decreasing and convex for $x \geqq 0$ and let (y) be given except in arrangement. Then $\Sigma_{i=1}^{n} f(|\, y_i - y_{i+1}\,|)$ (where $y_{n+1}=y_1$) is minimal if (and under some additional assumptions only if) (y) is arranged in circular symmetrical order.

The problem considered here arose in connection with a paper by B. Schwarz [2]. For the sake of completeness we start with a definition given there which differs slightly from the definition given in the book of Hardy, Littlewood and Pólya [1, Chapter X].

An ordered set $(a) = (a_1, \cdots, a_n)$ of n real numbers is called *symmetrically decreasing* if either

$$a_1 \leqq a_n \leqq a_2 \leqq a_{n-1} \leqq \cdots \leqq \leqq a_{[(n+2)/2]} \tag{1}$$

or

$$a_n \leqq a_1 \leqq a_{n-1} \leqq a_2 \leqq \cdots \leqq a_{[(n+1)/2]} \tag{2}$$

holds. For a given set $(y) = (y_1, \cdots, y_n)$ there exist, in general, two distinct *symmetrically decreasing rearrangements*. The rearrangement ordered as in (1) is denoted by $(y^-) = (y_1^-, \cdots, y_n^-)$ so that

$$y_1^- \leqq y_n^- \leqq y_2^- \leqq y_{n-1}^- \leqq \cdots \leqq y_{[(n+2)/2]}^- ; \tag{1'}$$

The other symmetrically decreasing rearrangement is denoted by

$$({}^-y) = ({}^-y_1, \cdots, {}^-y_n):$$

$${}^-y_n \leqq {}^-y_1 \leqq {}^-y_{n-1} \leqq {}^-y_2 \leqq \cdots \leqq {}^-y_{[(n+1)/2]}. \tag{2'}$$

We add the following definitions. A *circular rearrangement* of an ordered set $(y) = (y_1, \cdots, y_n)$ is a cyclic rearrangement of (y) or a cyclic rearrangement followed by inversion. For example, the circular rearrangements of the set $(1,2,3,4)$ are the sets

Received January 17, 1963 (Revised April 28, 1963).

* Sponsored by the Mathematics Research Center, United States Army under Contract No. DA-11-022-ORD-2059, University of Wisconsin, Madison.

$$(1,2,3,4),\quad (2,3,4,1),\quad (3,4,1,2),\quad (4,1,2,3),$$
$$(4,3,2,1),\quad (1,4,3,2),\quad (2,1,4,3),\quad (3,2,1,4).$$

An ordered set $(y)=(y_1,\cdots,y_n)$ of n real numbers is arranged in *circular symmetrical order* or is *of circular symmetry* if one of its circular rearrangements is symmetrically decreasing. It follows that the sets (y^-), (^-y) and (a), satisfying (1) or (2), are of circular symmetry, and so is the set $(b)=(b_1,\cdots,b_n)$ if either

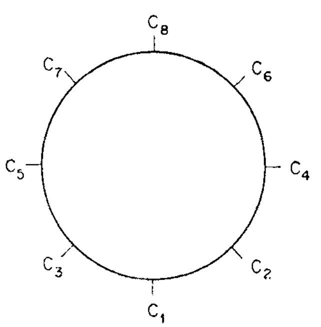
$$b_1 \leqq b_2 \leqq b_n \leqq b_3 \leqq b_{n-1} \leqq \cdots \leqq b_{[(n+3)/2]} \tag{3}$$

or

$$b_2 \leqq b_1 \leqq b_3 \leqq b_n \leqq b_4 \leqq b_{n-1} \leqq \cdots \leqq b_{[(n+4)/2]} \tag{4}$$

holds.

Sets of circular symmetry with a given set of n elements can be visualized as follows. Let c_i, $i=1,\cdots,n$ be the n given numbers and assume that $c_1 \leqq c_2 \leqq \cdots \leqq c_n$. Place these n numbers at n distinct points of a circle according to the following rule: c_1 may be put at any point; c_2 is a neighbor of c_1; c_3 is the other neighbor of c_1; c_4 is a neighbor of c_2; c_5 is a neighbor of c_3, etc. (We could also start with the largest element c_n.) The figure shows the construction for $n=8$. The sets of circular symmetry are obtained by starting at any point and going in clockwise or counter-clockwise direction over the circle.



Figure

We state now our result.

THEOREM. *Let the function $f(x)$ be non-decreasing and convex for $x \geqq 0$. Let the set (y) of n real numbers be given except in arrangement. Then*

$$\sum_{i=1}^{n} f(|y_i - y_{i+1}|) \tag{5}$$

(where $y_{n+1}=y_1$) is minimal if (y) is arranged in circular symmetrical order. Moreover, if the convexity of $f(x)$ is strict and no three elements of (y) have the

same value, then (5) *attains its minimum only if* (y) *is arranged in circular symmetrical order.*

Proof. (5) is clearly invariant under all circular rearrangements. The first assertion of the theorem is thus equivalent to

$$(6)\qquad \sum_{i=1}^{n} f(|y_i - y_{i+1}|) \geqq \sum_{i=1}^{n} f(|y_i^- - y_{i+1}^-|), \quad (y_{n+1} = y_1,\ y_{n+1}^- = y_1^-),$$

where $(y^-) = (y_1^-, \cdots, y_n^-)$ is the symmetrically decreasing arrangement of (y) satisfying (1′). To prove the second assertion of the theorem we have to show that equality holds in (6) only if (y) is of circular symmetry.

The proof proceeds by induction. For $n = 2$ and $n = 3$ every set is of circular symmetry and (5) is clearly invariant under all rearrangements, hence the theorem holds trivially for those n.

We now assume the validity of the theorem for sets of $n - 1$ numbers and show that this implies its validity for sets of n numbers. Let $(y) = (y_1, \cdots, y_n)$ be such a set. Without loss of generality we may assume that

$$(7)\qquad 0 = y_1 \leqq y_i, \qquad i = 2, \cdots, n.$$

(1′) and (7) imply that for $(y^-) = (y_1^-, \cdots, y_n^-)$

$$(8)\qquad 0 = y_1^- \leqq y_i^-, \qquad i = 2, \cdots, n.$$

We define the set $(x) = (x_1, \cdots, x_{n-1})$ of $n - 1$ numbers by

$$(9)\qquad x_i = y_{i+1}, \qquad i = 1, \cdots, n-1, \qquad (x_i \geqq 0).$$

Similarly, $(x') = (x_1', \cdots, x_{n-1}')$ is defined by

$$(10)\qquad x_i' = y_{i+1}^-, \qquad i = 1, \cdots, n-1, \qquad (x_i' \geqq 0).$$

(7) and (9) imply

$$(11)\qquad \sum_{i=1}^{n} f(|y_i - y_{i+1}|) = \sum_{i=1}^{n-1} f(|x_i - x_{i+1}|) - f(|x_{n-1} - x_1|) + f(y_2) + f(y_n)$$

$$= \sum_{i=1}^{n-1} f(|x_i - x_{i+1}|) + f(x_1) + f(x_{n-1}) - f(|x_1 - x_{n-1}|), (y_{n+1} = y_1 = 0;\ x_n = x_1).$$

Similarly, it follows from (8) and (10) that

$$\sum_{i=1}^{n} f(|y_i^- - y_{i+1}^-|) = \sum_{i=1}^{n-1} f(|x_i' - x_{i+1}'|) + f(x_1') + f(x_{n-1}') - f(|x_1' - x_{n-1}'|),$$

$$(y_{n+1}^- = y_1^- = 0;\ x_n' = x_1').$$

We define

$$g(s,t) = f(s) + f(t) - f(|s-t|); \qquad s \geqq 0, \qquad t \geqq 0. \tag{13}$$

(11)–(13) give

$$\begin{aligned}&\sum_{i=1}^{n} f(|y_i - y_{i+1}|) - \sum_{i=1}^{n} f(|y_i^- - y_{i+1}^-|)\\ &= \sum_{i=1}^{n-1} f(|x_i - x_{i+1}|) - \sum_{i=1}^{n-1} f(|x_i' - x_{i+1}'|) + g(x_1, x_{n-1}) - g(x_1', x_{n-1}').\end{aligned} \tag{14}$$

(7)–(10) imply that (x') is a rearrangement of (x). (1′) and (10) give

$$x_{n-1}' \leqq x_1' \leqq x_{n-2}' \leqq \cdots \leqq x_{[n/2]}'; \tag{15}$$

Hence, by (2), (x') is of circular symmetry (and indeed $(x') = ({}^-x)$) and it follows by the assumption of the induction that

$$\sum_{i=1}^{n-1} f(|x_i - x_{i+1}|) \geqq \sum_{i=1}^{n-1} f(|x_i' - x_{i+1}'|), \qquad (x_n = x_1, \quad x_n' = x_1'). \tag{16}$$

$f(x)$ is, by assumption, non-decreasing and convex for $x \geqq 0$. This implies (and is equivalent to the fact) that $g(s,t) = g(t,s)$, defined by (13), is a non-decreasing function of s and t, $s \geqq 0$, $t \geqq 0$. (15) shows that x_{n-1}' and x_1' are the two smallest numbers of (x). It follows that

$$g(x_1, x_{n-1}) \geqq g(x_1', x_{n-1}'). \tag{17}$$

(14), (16) and (17) imply (6) and we thus proved the first assertion of the theorem.

To prove the second assertion we assume that $(y) = (y_1, \cdots, y_n)$ takes every value at most twice. It follows that the same holds for $(x) = (x_1, \cdots, x_{n-1})$ and that this set takes the value 0 at most once. We also assume that the convexity of $f(x)$ is strict (and it thus follows that the non-decreasing function $f(x)$ is strictly increasing). This implies that $g(s,t)$ is a strictly increasing function of s and $t (s > 0,\ t > 0)$; $g(s,0) = g(0,t) = 0$; $g(s,t) > 0 (s > 0,\ t > 0)$.

Assume now that equality holds in (6); i.e.

$$\sum_{i=1}^{n} f(|y_i - y_{i+1}|) = \sum_{i=1}^{n} f(|y_i^- - y_{i+1}^-|), \quad (y_{n+1} = y_1 = y_{n+1}^- = y_1^- = 0). \tag{6′}$$

(6′), (14), (16) and (17) imply

$$\sum_{i=1}^{n-1} f(|x_i - x_{i+1}|) = \sum_{i=1}^{n-1} f(|x_i' - x_{i+1}'|), \qquad (x_n = x_1, \quad x_n' = x_1') \tag{16′}$$

and

$$g(x_1, x_{n-1}) = g(x_1', x_{n-1}'). \tag{17′}$$

As shown above, the rearrangement (x') of (x) is of circular symmetry. (16′) implies thus by the assumption of the induction, that (x) itself is of circular symmetry. x'_1 and x'_{n-1} are (by (15)) the two smallest numbers of (x'). (17′) and the properties of $g(s,t)$ imply therefore that either a) x_1 and x_{n-1} are both positive and are the two smallest numbers of (x); or b) either x_1 or x_{n-1} is equal to zero, but not both.

At this point it is convenient to use (9). The set of $n-1$ non-negative numbers $(y_2,\cdots,y_n)$ is of circular symmetry and does not take any values more than twice. In case a) y_2 and y_n are the smallest numbers of this set and they are positive. It follows that this set satisfies

$$(18) \qquad (0<)y_2 \leqq y_n \leqq y_3 \leqq y_{n-1} \leqq \cdots \leqq y_{[(n+3)/2}$$

or

$$(19) \qquad (0<)y_n \leqq y_2 \leqq y_{n-1} \leqq y_3 \leqq \cdots \leqq y_{[(n+2)/2]}.$$

(Note that we used the fact that the smallest value is taken at most twice.) In case b) we know that either $y_2=0$ or $y_n=0$ but not both, and that no other number of the set $(y_2,\cdots,y_n)$ of circular symmetry vanishes. Hence, one of the following four cases has to occur:

$$(18') \qquad (0=)y_2 \;<\; y_n \leqq y_3 \leqq y_{n-1} \leqq \cdots \leqq y_{[(n+3)/2]},$$

$$(20) \qquad (0=)y_2 \;<\; y_3 \leqq y_n \leqq y_4 \leqq \cdots \leqq y_{[(n+4)/2]},$$

$$(19)' \qquad (0=)y_n \;<\; y_2 \leqq y_{n-1} \leqq y_3 \leqq \cdots \leqq y_{[(n+2)/2]},$$

$$(21) \qquad (0=)y_n \;<\; y_{n-1} \leqq y_2 \leqq y_{n-2} \leqq \cdots \leqq y_{[(n+1)/2]}.$$

If we now complete the set $(y_2,\cdots,y_n)$ to a set of n elements by adding the term $y_1=0$ in the first place, then it is easily seen that in each case also the new set $(y_1,\cdots,y_n)$ is of circular symmetry. (For (18) and (18′) see (3); for (19) and (19′) see (1); for (20) see (4), and for (21) see (2)). This completes the proof of the theorem.

We now show that all assumptions of the theorem are necessary. Consider e.g. the sets

$$(22) \qquad (1,3,4,2) \text{ and } (1,2,3,4).$$

Here (and also in (23) and (24) below) the first set is of circular symmetry but the second is not. The sum (5) becomes, respectively, $2f(1)+2f(2)$ and $3f(1)+f(3)$. As there are increasing functions for which $2f(2)>f(1)+f(3)$ we cannot drop the assumption of convexity. Similarly, for the sets

$$(23) \qquad (1,3,5,4,2) \text{ and } (1,2,4,3,5),$$

The difference of the sums (5) becomes $f(2)-f(4)$, and we thus need the assumption that $f(x)$ is increasing. Using the sets (22) and the function $f(x)=x$, it follows that we have to assume strict convexity for the second part of the theorem. Finally for the sets

$$(1,2,3,4,2,2) \text{ and } (1,2,2,3,4,2), \tag{24}$$

the sums are, for any $f(x)$, equal; hence it is necessary to assume that no three elements of (y) have the same value.

An analysis of the proof shows that if in a set $(y) = (y_1, \cdots, y_n)$ one or several values are taken more than twice, then — under the stricter assumption on $f(x)$— (5) becomes minimal for exactly those rearrangements of (y) which are constructed by the following rule: Let $(z) = (z_1, \cdots, z_m)$, $m < n$, have the same elements as (y) except that, if (y) takes a value more than twice, then (z) contains only two elements having this value. Arrange (z) in circular symmetrical order and obtain the minimizing rearangements of (y) by inserting the additional $n - m$ elements next to elements having the same value.

The theorem yields the following result concerning linear arrangements.

COROLLARY. *Let the function $f(x)$ be non-decreasing and convex for $x \geqq 0$. Let the set (y) of n non-negative numbers be given except in arrangement. Then*

$$\sum_{i=1}^{n-1} [f(y_i) + f(y_{i+1}) - f(|y_i - y_{i+1}|)] \tag{25}$$

is maximal if (y) is arranged in symmetrical decreasing order. Moreover, if the convexity of $f(x)$ is strict and if all the elements of (y) are positive and no three of them have the same value, then this maximum is attained only if (y) is symmetrically decreasing.

Proof.

$$\begin{aligned} &\sum_{i=1}^{n-1} [f(y_i) + f(y_{i+1}) - f(|y_i - y_{i+1}|)] \\ &= 2\sum_{i=1}^{n} f(y_i) - \sum_{i=1}^{n} f(|y_i - y_{i-1}|) - [f(y_1) + f(y_n) - f(|y_1 - y_n|)]. \end{aligned} \tag{26}$$

As $\sum_{i=1}^{n} f(y_i)$ is invariant for all rearrangements of (y), (13) and (26) give

$$\begin{aligned} &\sum_{i=1}^{n-1} g(y_i^-, y_{i+1}^-) - \sum_{i=1}^{n-1} g(y_i, y_{i+1}) \\ &= \sum_{i=1}^{n} f(|y_i - y_{i+1}|) - \sum_{i=1}^{n} f(|y_i^- - y_{i+1}^-|) + g(y_1, y_n) - g(y_1^- y_n^-), \end{aligned} \tag{27}$$

y_1^- and y_n^- are (by (1′)) the two smallest elements of (y); hence

(28) $$g(y_1, y_n) \geqq g(y_1^-, y_n^-).$$

(6), (28) and (27) imply

(29) $$\sum_{i=1}^{n-1} g(y_i, y_{i+1}) \leqq \sum_{i=1}^{n-1} g(y_i^-, y_{i+1}^-) = \sum_{i=1}^{n-1} g({}^-y_i, {}^-y_{i+1}),$$

and thus we have proved the first assertion of the corollary.

If $y_i > 0$, $i = 1, \cdots, n$, then it follows from the stricter assumptions on $f(x)$ that

(28′) $$g(y_1, y_n) = g(y_1^-, y_n^-)$$

implies that y_1 and y_n are the two smallest elements of (y). By the second part of the theorem it also follows that

(6′) $$\sum_{i=1}^{n} f(|y_i - y_{i+1}|) = \sum_{i=1}^{n} f(|y_i^- - y_{i+1}^-|)$$

implies that (y) is of circular symmetry. Under the assumptions of the second part of the corollary it thus follows (using also (6), (28) and (27)) that (25) becomes maximal only if (y) is of circular symmetry and if y_1 and y_n are its two smallest elements. But this implies $(y) = (y^-)$ or $(y) = ({}^-y)$ and the proof of the corollary is complete.

We now show also for the corollary that all its assumptions are necessary. Note that in (22) and (24) (and also in (30) and (31) below) the first set is symmetrically decreasing but the second is not. For the sets (22) the sum (25) becomes respectively, $-f(2) + 2f(3) + 2f(4)$ and $-2f(1) + 2f(2) + 2f(3) + f(4)$. As there are increasing functions for which $f(4) + 2f(1) < 3f(2)$ we cannot drop the assumption of convexity. For the sets

(30) $$(1, 3, 2) \text{ and } (3, 1, 2)$$

the difference of the sums (25) becomes $f(3) - f(1)$, and we thus need the assumption that $f(x)$ is increasing. Using again the sets (22) and the function $f(x) = x$, it follows that we have to assume strict convexity for the second part of the corollary. The sets (24) prove again that it is necessary to assume that no three elements of (y) have the same value and, finally, the sets

(31) $$(0, 2, 1) \text{ and } (0, 1, 2)$$

show that we have to assume $y_i > 0$ for the second part of the corollary.

References

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